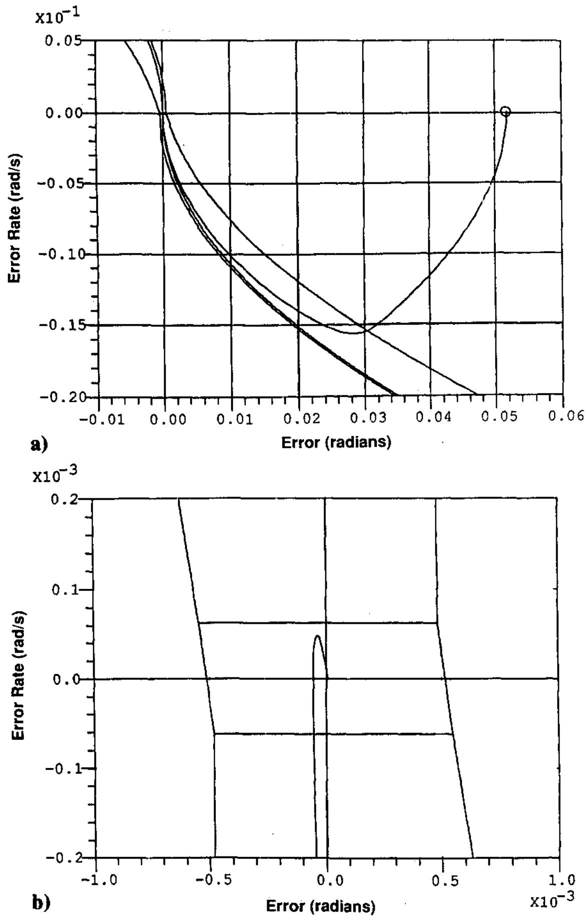

Fig. 4 Example of error trajectory in the phase plane.

Example

Simulation analysis played a key role in the development of this control law, being critical for quantitive evaluation of control system performance as a function of design parameters for any specific application. In this example, the control system is required to null pointing errors while tracking a target whose motion is given by

$$\theta_{CMD}(t) = \theta_p + \theta_v(t) + \theta_a(t^2/2) + \theta_j(t^3/6) \tag{15}$$

where θ_p, θ_v, θ_a, and θ_j are randomly chosen within prescribed bounds. To meet the tracking requirement, the linear control law needed two integrators. A second-order, high-frequency attenuation was included in the linear controller. Thus, $K(s)$ in Eq. (1) was selected as

$$K(s) = k(s^2 + as + b)/[s^2(s^2 + cs + d)] \tag{16}$$

The values of the parameters of the design model for this example are $J = 70$ kgm^2 and $T_L = 0.4$ Nm. The design parameters chosen for the control law are 1) $t_L = T_L$, 2) the linear control bandwidth = 8 rad/s, and, finally, 3) the height of R_1 is 1/20th of that shown in Fig. 3, and the switching and transition curves are chosen to pass through the vertices of R_1.

The performance of the control law is demonstrated via simulation of a sample target trajectory. The error trajectory in the phase plane for this case is shown in Fig. 4a, and Fig. 4b shows an enlargement of the phase plane near R_1. In both parts, the switching and transition curves are shown, and in Fig. 4a the switching curve for time-optimal control to the origin is also shown. Figure 5 shows the profile of the control torque. It consists of a continuous startup, a single period of saturation, a transition segment, and at about 5 s, a smooth change from the transition control to the linear control law.

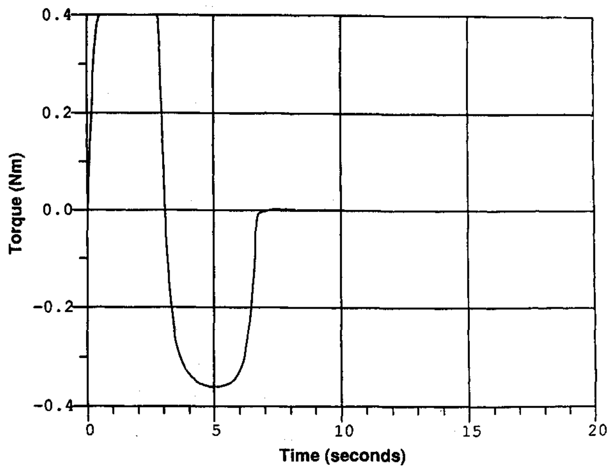

Fig. 5 Example of torque profile.

Conclusions

A blend of linear and time-optimal control is capable of providing acquisition, tracking, and pointing with high accuracy and short response times. Smoothness in a blended feedback control law can be obtained for low-order design models, and simulation is a necessary ingredient in the derivation of such a controller.

References

[1]Oldenburger, R., *Optimal and Self-Optimizing Control,* M.I.T. Press, Cambridge, MA, 1966.

[2]Harvey, C. A., "Synthesis of Time-Optimal Control for Linear Processes," *Journal of Mathematical Analysis and Applications,* Vol. 10, 1965, pp. 334–341.

Projective Formulation of Maggi's Method for Nonholonomic Systems Analysis

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Introduction

THOUGH first published late in the last century, Maggi's equations, until quite recently, have not been applicable in practice and primarily have been of academic interest only.[1] In a recent paper,[2] however, Papastavridis shows a variety of practical implementations of Maggi's approach as an efficient technique for the dynamic analysis of systems subject to nonholonomic constraints. In this Note, a geometrical insight into Maggi's method is presented. Based on the concept of the projection method,[3] the present formulation consists of the partition of the system's configuration space into the orthogonal and tangent subspaces; the orthogonal subspace is spanned by the constraint vectors, and the tangent subspace complements the orthogonal subspace in the configuration space. The projection of the initial (multiplier-containing) dynamical equations into the tangent subspace gives the con-

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straint reaction-free equations of motion, whereas the orthogonal projection determines the constraint reactions. Implementation of independent quasivelocities, which is the crux of Maggi's method, also transforms the reaction-free equations of motion to reduced-dimension equations of motion (Maggi's equations). The application of tensor algebra analysis and matrix notation yields a mathematical formulation that is commendably concise. The well-known knife-edge problem is studied as an illustration.

Definitions and Background

The starting point of Maggi's approach is a mechanical system characterized by n generalized coordinates $q=[q_1,\dots,q_n]^T$ and subjected to m nonholonomic constraints $\varphi=[\varphi_1,\dots,\varphi_m]^T$. Assuming for generality that the initial equations of motion of the unconstrained system are derived in quasivelocities $v=[v_1,\dots,v_n]^T$, the governing equations of the constrained motion can be written as follows:

$$M\dot{v} = h + C_\lambda^T\lambda \tag{1}$$

$$\dot{q} = Av + a \tag{2}$$

$$C_\lambda\dot{v} + c_\lambda = 0 \tag{3}$$

where $M(q,t)$ is an $n\times n$ symmetric positive-definite (metric tensor) matrix; $A(q,t)$ is an $n\times n$ invertible matrix; $h(v,q,t)$ and $a(q,t)$ are $n\times 1$ matrices (vectors) of covariant and contravariant components, respectively; $C_\lambda(v,q,t)$ is an $m\times n$ constraint matrix, and the so-called constraint vectors[4] are contained in C_λ^T as columns (covariant components); $c_\lambda(v,q,t)$ is an $m\times 1$ matrix; $\lambda=[\lambda_1,\dots,\lambda_m]^T$ are Lagrangian multipliers; and t is the time. The vectors v and q are represented by contravariant components, and A is the transformation matrix $e_v=A^Te_q$, where $e_v=[e_{v1},\dots,e_{vn}]^T$ and $e_q=[e_{q1},\dots,e_{qn}]^T$ are the covariant base vectors. Moreover, v defined in Eq. (2) should actually be called a vector of kinematical parameters as its components may be either quasivelocities and/or generalized velocities.

Equation (3) represents the constraint equations transformed into the dynamical (second-order kinematical) form by differentiating with respect to time the initial nonholonomic constraint equations, which can be represented in both the general and linear forms, i.e.,

$$\varphi(v,q,t) = 0 \tag{4a}$$

$$Dv + d = 0 \tag{4b}$$

where $D(q,t)$ and $d(q,t)$ are $m\times n$ and $m\times 1$ matrices, respectively. Accordingly, $C_\lambda=\partial\varphi/\partial v$ or D, and $c_\lambda=(\partial\varphi/\partial q)(AV+a)+\partial\varphi/\partial t$ or $\dot{D}v+\dot{d}$.

A class of ideal constraints is considered in this Note. In the mathematical sense, the ideal constraint reactions are collinear with the constraint vectors[1,4]; that is, the ith constraint reaction is represented in the base e_v as follows:

$$r_i = C_{\lambda i}^T\lambda_i \tag{5}$$

where $C_{\lambda i}^T$ denotes the ith column of C_λ^T, and $r=\mathrm{sum}(r_i)=C_\lambda^T\lambda$ is the vector of the constraining forces represented in Eq. (1).

Projection Method Reformulation

Denote the constraint vectors by $e_\lambda=[e_{\lambda 1},\dots,e_{\lambda m}]^T$, which are represented in the base e_v by covariant components contained in C_λ^T as columns. Since e_λ are independent [$\mathrm{rank}(C_\lambda)=\max$], a set of $k=n-m$ independent vectors $e_\tau=[e_{\tau 1},\dots,e_{\tau k}]^T$ can be chosen so as to be orthogonal to e_λ. Assuming that the vectors e_τ are represented in the base e_v by contravariant components contained in D^T as columns, where $D(q,t)$ is a $k\times n$ matrix of maximal rank, the orthogonality condition (dot products $e_\tau e_\lambda^T=0$) is equivalent to

$$C_\tau C_\lambda^T = 0 \tag{6}$$

In other words, C_τ is an orthogonal complement of C_λ in the n space.

The vectors $e'=[e_\lambda^T,e_\tau^T]^T$, being linearly independent, form a new base in the system's configuration space, and the following formula for the transformation between the covariant bases can be written as

$$e' = \begin{bmatrix} e_\lambda \\ e_\tau \end{bmatrix} = \begin{bmatrix} C_\lambda M^{-1} \\ C_\tau \end{bmatrix} e_v = Te_v \tag{7}$$

Since Eq. (1) is expressed in the base e_v by covariant components, the covariant representation in the base e' is equivalent to the left-sided multiplication of Eq. (1) by T. Denoting that e_λ and e_τ span complementary orthogonal and tangent subspaces, the resultant dynamic equations in the base e' can be decomposed (projected into the two subspaces) as follows:

$$C_\lambda\dot{v} = C_\lambda M^{-1}h + C_\lambda M^{-1}C_\lambda^T\lambda \tag{8a}$$

$$C_\tau M\dot{v} = C_\tau h \tag{8b}$$

The tangent projection (8b), combined with Eqs. (3) and (4), gives the reaction-free governing equations:

$$TM\dot{v} = h' \tag{9a}$$

$$\dot{q} = Av + a \tag{9b}$$

where $h'=[-c_\lambda^T,(C_\tau h)^T]^T$, and the dimension of Eqs. (9) is $2n$, as compared with the dimension $2n+m$ of Eqs. (1-3). It is worth noting that Eq. (8b) is equivalent conceptually to the results obtained by Hemami and Weimer[5] by using the orthogonal complement method.

The orthogonal projection (8a) may serve for determination of λ and, using Eq. (5), r_i; namely,

$$\lambda = -M_\lambda^{-1}(c_\lambda + C_\lambda M^{-1}h) = \lambda(v,q,t) \tag{10}$$

where M_λ is an $m\times m$ metric tensor matrix of the orthogonal subspace, which can be defined as an element of the metric tensor matrix of the base e', i.e.,

$$M' = TMT^T = \begin{bmatrix} C_\lambda M^{-1}C_\lambda^T & 0 \\ 0 & C_\tau MC_\tau^T \end{bmatrix} = \begin{bmatrix} M_\lambda & 0 \\ 0 & M_\tau \end{bmatrix} \tag{11}$$

and M_τ is the metric tensor matrix of the tangent subspace.

Maggi's Equations

The essence of Maggi's method (originally for linear constraints only[1,2]) lies in the treatment of Eqs. (4) as a set of m null quasivelocities. Hence, $k=n-m$ independent quasivelocities $u=[u_1,\dots,u_k]^T$ exist such that the transformation between $[0^T,u^T]^T$ (0 denotes here the $m\times 1$ matrix with null entries) and v can be stated explicitly. In the case of linear constraints, u can be defined conveniently as represented only in the tangent subspace, i.e.,

$$M'\begin{bmatrix} 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ M_\tau u \end{bmatrix} = \begin{bmatrix} Dv + d \\ C_\tau Mv \end{bmatrix} = TMv + \begin{bmatrix} d \\ 0 \end{bmatrix} \tag{12}$$

where C_τ is an orthogonal complement of D ($C_\tau D^T=0$), and D, d, M', and M_τ are defined in Eqs. (4) and (11), respectively. In the meaning of Eq. (12), u is the projection of v into the tangent subspace.

From Eq. (12) the following relation can be found:

$$v = C_\tau^Tu - M^{-1}D^TM_\lambda^{-1}d \tag{13}$$

Now, since the constraint equations expressed in u are satisfied in principle, the introduction of Eq. (13) into Eqs. (9) yields the first m of the equations to identity and the remaining $k+n$ equations to the following form of Maggi's equations:

$$M_\tau \dot{u} = h_\tau \tag{14a}$$

$$\dot{q} = AC_\tau^T u + \tilde{a} \tag{14b}$$

where $h_\tau(u,q,t) = C_\tau(h - M\dot{C}_\tau u) - (C_\tau M)^{\cdot} M^{-1} D^T M_\lambda^{-1} d$, and $\tilde{a}(q,t) = a - AM^{-1}D^T M_\lambda^{-1} d$.

Unfortunately, for nonlinear nonholonomic constraints $\varphi(v,q,t)=0$, the definition of u is not so evident, if feasible in practice at all. Hence, for this case, the (somewhat more general) projection method reported earlier is recommended.

Example

Consider a well-known problem of knife-edged motion.[1,2] Assuming that the knife's blade remains perpendicular to the motion plane, the governing equations, corresponding to Eqs. (1-3), are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} mv_3v_2 + Q_1\cos q_3 + Q_2 \sin q_3 \\ -mv_3v_1 - Q_1 \sin q_3 + Q_2 \cos q_3 \\ Q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix} \lambda \tag{15}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{16}$$

$$\dot{v}_2 + s\dot{v}_3 = [0,\ 1,\ s] \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = C_\lambda \dot{v} = 0 \tag{17}$$

where m and J are the mass and moment of inertia, respectively; q_1 and q_2 the position coordinates of C; q_3 the angle as shown in Fig. 1; v_1 and v_2 the projections of the linear velocity of C onto the axes of the blade-fixed reference frame; and v_3 the knife's angular velocity (note that v_1 and v_2 are quasi-velocities, whereas v_3 is a generalized velocity). The external forces applied to the body are represented by the resultant force components Q_1 and Q_2 relative to inertial frame and the torque Q_3 of the forces in relation to the mass center C.

Equation (17) expresses the dynamical form of the nonholonomic constraint imposed. In its original form, the constraint equation is

$$\varphi = v_2 + sv_3 = 0 \tag{18}$$

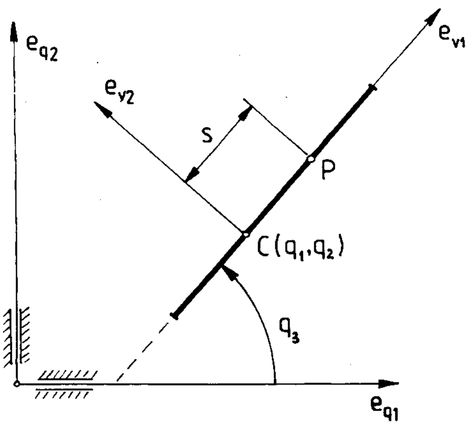

Fig. 1 Knife-edge problem.

which represents the demand of collinearity of the vector of velocity of the contact point P to the knife's edge, and s is the distance from C to P.

Following the mathematical formulation just reported, the matrix C_τ can be constructed as

$$C_\tau = \begin{bmatrix} 0 & -s & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{19}$$

and the reaction-free dynamic equations [referring to Eq. (9a)] are

$$\begin{bmatrix} 0 & 1 & s \\ 0 & -ms & J \\ m & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ s(mv_1v_3 + Q_1 \sin q_3 - Q_2 \cos q_3) + Q_3 \\ mv_2v_3 + Q_1 \cos q_3 + Q_2 \sin q_3 \end{bmatrix} \tag{20}$$

Then, the multiplier λ and the constraint reaction r are

$$\lambda = -\mu F_\lambda \tag{21}$$

$$r = -[0,\ 1,\ s]^T \mu F_\lambda \tag{22}$$

where

$$\mu = J/(J + ms^2)$$

and

$$F_\lambda = -mv_1v_3 + Q_1 \sin q_3 + Q_3 ms/J$$

Now, let us introduce the independent quasivelocities. According to the definition given in Eq. (12), they can be stated as follows:

$$\begin{bmatrix} \dfrac{J+ms^2}{mJ} & 0 & 0 \\ 0 & J+ms^2 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & s \\ 0 & -ms & J \\ m & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{23}$$

which leads to the inverse relation [corresponding to Eq. (13)]

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -s & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{24}$$

The final reduced-dimension governing equations in the form of Eqs. (14) are then as follows:

$$\begin{bmatrix} J+ms^2 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} s(mu_1u_2 + Q_1 \sin q_3 - Q_2 \cos q_3) + Q_3 \\ -msu_1^2 + Q_1 \cos q_3 + Q_2 \sin q_3 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} s \sin q_3 & \cos q_3 \\ -s \cos q_3 & \sin q_3 \\ 1 & 0 \end{bmatrix} \tag{26}$$

Using Eq. (24), λ and r defined in Eqs. (21) and (22) can be expressed as functions of u, q, and t.

Conclusions

This Note presents a projective interpretation of the well-known Maggi's approach to dynamic analysis of nonholonomic systems. Both linear and nonlinear constraint cases were dealt within a unified treatment. The projection method used does not require the use of variational principle. Instead, the language of vector spaces is used, and tensor algebra analysis is applied to clarify the mathematical transformations and render the formulation compact.

Acknowledgments

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References

[1]Nejmark, J. I., and Fufajev, N. A., *Dynamics of Nonholonomic Systems*, Nauka, Moscow (in Russian), 1967 (in English: *Translations of Mathematical Monographs*, Vol. 33, American Mathematical Society, Providence, RI, 1972).

[2]Papastavridis, J. G., "Maggi's Equations of Motion and the Determination of Constraint Reactions," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 2, 1990, pp. 213-220.

[3]Scott, D., "Can a Projection Method for Obtaining Equations of Motion Compete with Lagrange's Equations?" *American Journal of Physics*, Vol. 56, No. 5, 1988, pp. 451-456.

[4]Kamman, J. W., and Huston R. L., "Dynamics of Constrained Multibody Systems," *Journal of Applied Mechanics*, Vol. 51, No. 4, 1984, pp. 899-903.

[5]Hemami, H., and Weimer, F. C., "Modelling of Nonholonomic Dynamic Systems with Applications," *Journal of Applied Mechanics*, Vol. 48, No. 1, 1981, pp. 177-182.

Probing Behavior in Certain Optimal Perturbation Control Laws

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Introduction

THE results here are only formal in nature because they are based on optimal control law approximations that are derived with a dynamic programming analysis in which neglected error terms are assumed without proof to be sufficiently well behaved that all quantities that appear to be of negligible orders of magnitude actually are so in some precise and appropriate sense. Unless otherwise stated in what follows, lower case letters denote (real) column vectors or scalars. Matrices are denoted by capital Roman letters. Capital Greek letters, however, denote three-way matrices, and the following definitions are adopted for such an object Ω, with matrices A and B of compatible dimensions, and with repeated indices denoting summation:

$$(\Omega')_{ijk} = \Omega_{jki} \qquad \text{(three-way matrix)}$$

$$(A\Omega)_{ijk} = A_{i\sigma}\Omega_{\sigma jk}$$

$$\text{and} \quad (\Omega B)_{ijk} = \Omega_{ij\sigma}B_{\sigma k} \qquad \text{(three-way matrices)}$$

$$\text{tr}(\Omega)_i = \Omega_{\sigma i\sigma} \qquad \text{(column vector)}$$

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Expressions denoting ordinary differential equations with white noise terms should be understood as the formally corresponding stochastic differential equations in the sense of Ito if a rigorous interpretation is desired.

The topic treated here arises from a stochastic optimal control problem with multivariate dynamics of the general form

$$\dot{y} = f(y, v, t, \omega);\ y(t_0) \ \text{normal with mean} \ \bar{y}_0 \ \text{a priori}$$

measurements at each time instant t of the form

$$\zeta = g(y, v, t, n)$$

and scalar performance criterion (to be minimized over control laws for v as a function of previous ζ) of the form

$$J = \mathcal{E}\{\psi[y(t_f)] + \int_{t_0}^{t_f} \lambda(y, v, t)\, dt\}$$

where t_0 and t_f are specified a priori, ω and n are vector white noise "processes," and $\mathcal{E}$ denotes expectation. A related deterministic problem, which is easier to solve, can be defined by making $\omega = 0$, $n = 0$, and $y(t_0) = \bar{y}_0$, and its solution specifies nominal time histories $\bar{y}(t)$, $\bar{v}(t)$, $\bar{\zeta}(t)$, and performance $\bar{J}$. The original problem can then be solved by adding to $\bar{v}$ the perturbation control law for $v - \bar{v}$ that minimizes $J - \bar{J}$. Taylor-series expansions of f, g, ψ, and λ can typically be used to express the dynamics, measurements, and criterion of this optimal perturbation control problem in terms of the noise-induced perturbation variables $y - \bar{y}$, $v - \bar{v}$, $\zeta - \bar{\zeta}$, $J - \bar{J}$, ω, and n. Reference will be made to two asymptotic approximations of such a problem for small perturbations. One is a problem of the standard linear-quadratic-Gaussian form that results from truncating the previous expansions at the lowest nontrivial orders.[1] The other is the more accurate approximation obtained by carrying out these expansions to one higher order, which typically introduces quadratic terms in the dynamics and measurements and cubic terms in the criterion. The units can often be scaled so that the perturbation variables are numerically of order unity and the coefficients of the higher degree terms added in the latter approximation become the quantities that are relatively small, say of order h. Solutions to perturbation control problems that are normalized in this way have been approximated with errors of order h^2 when the other problem parameters and the inverses of two of them (namely, the matrices B and R below) are all of order unity.[2,3] However, R^{-1} is large in many cases of interest because the measurements are highly accurate, and such normalized problems with large R^{-1}, subject to certain conditions, are the topic of this Note.

Class of Perturbation Control Problems

The first of the conditions just mentioned is that the perturbation state can be partitioned as

$$\begin{bmatrix} x \\ \theta \end{bmatrix}$$

such that the linear-quadratic-Gaussian approximation has dynamics of the form

$$\begin{cases} \dot{x} = Fx + Gu + w; & x(t_0) \text{ is normal } (\bar{x}_0, P_0) \text{ a priori} \quad (1) \\ \dot{\theta} = hw_2; & \theta(t_0) \text{ is normal } (0, L_0) \text{ a priori} \quad (2) \end{cases}$$

where u is the perturbation control, perturbation measurements of the form

$$z = Hx + v \qquad (3)$$